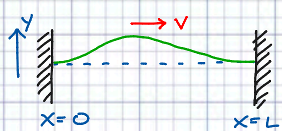
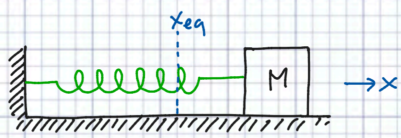
# Vector Calc

- In physics we talk about lots of quantities – temperature, the acceleration due to gravity, electric fields – that change from place to place, or over time.

- We express these changes using derivatives, and then we work out laws & equations that tell us how the the quantities behave.

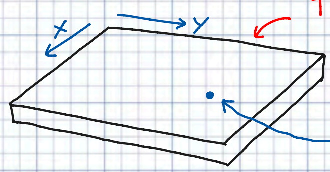
- When the rule is for something like the displacement of a string from equilibrium, or the oscillation of a mass on a string, the stuff you learned in single variable calc is sufficient:




$$\frac{d^2y(x,t)}{dx^2} - \frac{1}{v^2}\frac{d^2y(x,t)}{dt^2} = 0 \qquad M\frac{d^2x(t)}{dt^2} = -k_x\left(x(t) - x_{eq}\right)$$

- But what about quantities where we need two or more coordinates to specify the position? For instance, the temperature @ a pt. (x,y) on a rectangular plate w/ one edge @ fixed T?

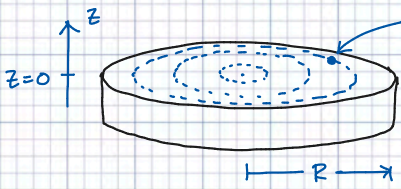



This end (x=0) held @ constant temperature, starting @ time t=0.

What is temp T @ this point 10 seconds later?

- To write the eqn governing heat flow & temp. $T(x,y,t)$, need to describe changes in x & y directions. How?

- And what about something like a circular drum head, where it makes sense to use polar coords $(\rho, \phi)$?




How does pt. @ $\rho = R/2$, $\phi = \pi/4$ move up & down? What is height above / below eq. @ $t = 5s$?

Is there something like the string eqn? How do $d/d\rho$ & $d/d\phi$ show up?

- Also, what about <u>vector</u> quantities like the Electric field $\vec{E}$?

- In this section we're going to generalize what you know about derivatives to functions (both scalar and vector!) of multiple variables, and in different coordinate systems.

- We'll see that there's more than one sort of "derivative," with various ways to combine them. And each one has its own "FUNDAMENTAL THEOREM," its version of

$$\int_A^B dx \, \frac{df(x)}{dx} = f(B) - f(A)$$

"INTEGRATION IS ANTI-DIFFERENTIATION"

- Let's get started.

## THE GRADIENT

- When we studied OCS, we talked about the DIFFERENTIAL of a function. It tells us how much a function changes when we change its arguments.

- So if we have a SCALAR! function $f(x,y,z)$, the difference b/t $f$ @ two infinitesimally nearby points is:

$$df(x,y,z) = f(x+dx, y+dy, z+dz) - f(x,y,z)$$
$$= \frac{df}{dx}dx + \frac{df}{dy}dy + \frac{df}{dz}dz$$

Yes, this should be $\frac{d f(x,y,z)}{dz}$

$$\Rightarrow df = \frac{df}{dx}dx + \frac{df}{dy}dy + \frac{df}{dz}dz$$

- There is lots of info here. Suppose I start @ $(1,7,-3)$ and move a small distance '$dl$' in the $\frac{1}{\sqrt{2}}(\hat{x}-\hat{y})$ direction:

$$d\vec{l} = dl\frac{1}{\sqrt{2}}(\hat{x}-\hat{y}) = \overbrace{\left(\frac{1}{\sqrt{2}}dl\right)}^{dx}\hat{x} + \overbrace{\left(-\frac{1}{\sqrt{2}}dl\right)}^{dy}\hat{y}$$

No $\hat{z}$ displ.

$$\hookrightarrow df = \underbrace{\left.\frac{df}{dx}\right|_{(1,7,-3)}}_{\text{Rate } f \text{ changes when moving in x-dir. @ pt. } (1,7,-3)} \times \underbrace{\frac{1}{\sqrt{2}}dl}_{\text{Displacement in x-dir.}} + \underbrace{\left.\frac{df}{dy}\right|_{(1,7,-3)}}_{\text{Rate } f \text{ changes in y-dir. @ } (1,7,-3)} \times \underbrace{\left(-\frac{1}{\sqrt{2}}dl\right)}_{\text{Displacement in y-dir.}}$$

- So if you can take each derivative, you can tell me the rate a function changes when you move in <u>any</u> direction.

- Another way to think of this is to imagine you start @ some pt. 'P' & move an infinitesimal distance $dl$ in a direction specified by some <u>UNIT</u> <u>VECTOR</u> $\hat{u}$. To make things easy, let's just look @ a function of $x$ & $y$.

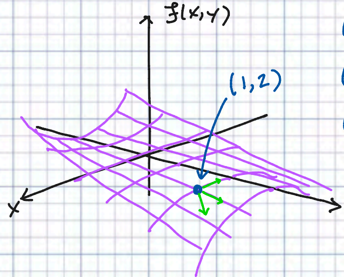
$$\left.df\right|_P = \left.\frac{df}{dx}\right|_P \times (u_x\,dl) + \left.\frac{df}{dy}\right|_P \times (u_y\,dl) \longleftarrow d\vec{l} = dl\underbrace{(u_x\hat{x}+u_y\hat{y})}_{\hat{u}}$$

$$\hookrightarrow \left.\frac{df}{dl}\right|_P = \left.\frac{df}{dx}\right|_P u_x + \left.\frac{df}{dy}\right|_P u_y \longleftarrow$$

"Derivative of $f$ in the direction $\hat{u}$ @ pt. P"

- Maybe you've seen this called a "DIRECTIONAL DERIVATIVE."

Ex| $f(x,y) = -x^2 y$     What is dir. derivative @ $x=1$, $y=2$ in




(a) $\hat{u} = -\hat{x}$ dir.

(b) $\hat{u} = \hat{y}$ dir.

(c) $\hat{u} = \frac{1}{\sqrt{2}}(\hat{x}+\hat{y})$ dir.

$$df = \frac{df}{dx}dx + \frac{df}{dy}dy = -2xy\,dx - x^2 dy$$

(a) $d\vec{l} = dl\,\hat{u} = -dl\,\hat{x} \Rightarrow dx = -dl \quad dy = 0$

$df(1,2) = -2\cdot 1\cdot 2\,dx - 1^2\cdot dy = -4\,dx - dy = +4\,dl$

$\Rightarrow \left.\frac{df}{dl}\right|_{(1,2)} = 4$ in the $-\hat{x}$ dir. ← Makes sense! $f(x,y)$ gets bigger if we start @ $(1,2)$ & move in $-\hat{x}$ dir.

(b) $d\vec{l} = dl\,\hat{u} = dl\,\hat{y} \Rightarrow dx = 0 \quad dy = dl$

$df(1,2) = -4\,dx - dy = -dl$

$\Rightarrow \left.\frac{df}{dl}\right|_{(1,2)} = -1$ in the $+\hat{y}$ dir.

(c) $d\vec{l} = dl\,\hat{u} = \frac{dl}{\sqrt{2}}\hat{x} + \frac{dl}{\sqrt{2}}\hat{y} \Rightarrow dx = dy = \frac{dl}{\sqrt{2}}$

$df(1,2) = -\frac{4}{\sqrt{2}}dl - \frac{1}{\sqrt{2}}dl = -\frac{5}{\sqrt{2}}dl$

$\Rightarrow \left.\frac{df}{dl}\right|_{(1,2)} = -\frac{5}{\sqrt{2}}$ in the $\frac{1}{\sqrt{2}}(\hat{x}+\hat{y})$ direction.

- Now, theres an easier way to get @ this info. Humor me for a moment. To move to an infinitesimally nearby point, we follow some displacement vector $d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$.

- What could we combine with this vector to get the scalar quantity $df(x,y,z)$? That is:

$$df = \frac{df}{dx}dx + \frac{df}{dy}dy + \frac{df}{dz}dz = (dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}) \cdot (\vec{?})$$

- Ah! We could dot $d\vec{l}$ into some vector thing whose $\hat{x}, \hat{y}$, & $\hat{z}$ components are the derivatives of $f(x,y,z)$!

$$\boxed{df = (dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}) \cdot \underbrace{\left(\hat{x}\frac{df}{dx} + \hat{y}\frac{df}{dy} + \hat{z}\frac{df}{dz}\right)}_{\text{The GRADIENT of } f(x,y,z)}}$$

- A shorter way to write this is $df = d\vec{l} \cdot \vec{\nabla} f$
- The notation '$\vec{\nabla} f$' is due to MAXWELL.
- This makes it really easy to get the directional derivative. Given some direction $\hat{u}$:

$$\frac{df}{dl} = \underbrace{\hat{u}}_{\text{Unit vector}} \cdot \overbrace{(\vec{\nabla} f)}^{\text{Grad } f}$$

Pick a direction $\hat{u}$ & eval. this @ some specific pt. $(x=\ldots, y=\ldots, z=\ldots)$ to get dir. der. in that direction @ that point.

Ex| $f(x,y,z) = A \times e^{-(x^2+y^2)} z^2$ w/ $A$ = constant

$$\vec{\nabla} f = -2xz^2 A e^{-(x^2+y^2)}\hat{x} - 2yz^2 A e^{-(x^2+y^2)} + 2zAe^{-(x^2+y^2)}\hat{z}$$

$\hat{u} = \hat{x}$ @ $P = (4,0,2)$?

$$\hat{u} \cdot \vec{\nabla} f = \hat{x} \cdot \vec{\nabla} f = -2xz^2 A e^{-(x^2+y^2)}$$

$$(\hat{u} \cdot \vec{\nabla} f)\Big|_{(4,0,2)} = -32\,A\,e^{-16}$$

- So the gradient contains all the same info as the differential. But since it's a vector, it also tells us other things.

- For instance, since $d\vec{l}$ & $\vec{\nabla} f$ are both vectors:

$$\underbrace{df}_{\text{CHANGE in } f} = d\vec{l} \cdot \vec{\nabla} f = \underbrace{|d\vec{l}|}_{\substack{\text{Mag. of} \\ d\vec{l}}} \, \underbrace{|\vec{\nabla} f|}_{\substack{\text{Mag. of} \\ |\vec{\nabla} f}} \cos\psi \leftarrow \measuredangle \text{ b/t } d\vec{l} \text{ \& } \vec{\nabla} f$$

- So the change in $f$ depends on the direction you go, and here we see that show up as a factor of $\cos\psi$. This is largest when $\psi = 0$ — when $d\vec{l}$ & $\vec{\nabla} f$ point in the <u>same direction</u>. In other words, @ any pt. we get the <u>LARGEST</u> $df$ by moving in the direction pointed out by $\vec{\nabla} f$!

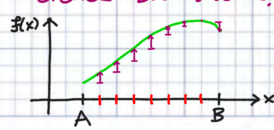
- The gradient of a function contains all the info about how it changes in each direction. If we move a small distance $d\vec{l}$, the change is $df = d\vec{l} \cdot (\vec{\nabla} f)$.

- Now, forget gradients for a moment & go back to single variable calc. One of the most important results you learned there was a relationship b/t integrals & derivatives that you probably called "THE FUNDAMENTAL THM OF CALCULUS".

$$\int_A^B \underbrace{dx \frac{df(x)}{dx}}_{df(x)} = f(B) - f(A)$$

$$\hookrightarrow \int_A^B df(x) = f(B) - f(A)$$

Nothing too deep! Add up all the <u>changes</u> in $f$ b/t A & B, get difference b/t $f(B)$ & $f(A)$:



- So a less exciting but totally equivalent statement of this F.T. is:

$$\int_A^B df = f(B) - f(A)$$

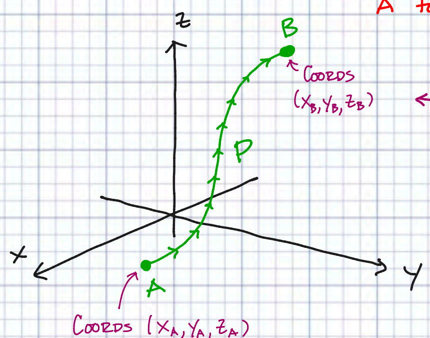
- But nothing about that refers to an x-axis or a straight line! It's just: "Add up all the changes in $f$ & you get the total change in $f$", right?

- Well if I move a small (infinitesimal) displacement $d\vec{l}$, a function $f(x,y,z)$ changes by $d\vec{l} \cdot \vec{\nabla} f$, so

$$\int_A^B df = \int_{A,\,P}^{B} d\vec{l} \cdot (\vec{\nabla} f) = f(B) - f(A)$$

Displ. from pt. to pt. along P

Whatever path P I follow from A to B




FUNDAMENTAL THEOREM OF GRADIENTS

Add up all the changes in $f$ along a path from A to B, you get the difference b/t $f(B)$ & $f(A)$.

- This is really the same thing as the F.T. you're already familiar with – sum of changes = total change. We're just adding them up along some path that maybe isn't a straight line like the x-axis!

$\boxed{\text{Ex}}$ Force $\vec{F}$ applied along some path from A to B? Work is:

$$W_P = \int_A^B d\vec{l} \cdot \vec{F} \quad \begin{cases} \text{Conservative Force: } \vec{F} = -\vec{\nabla} U \Rightarrow \underbrace{W_P = -\int_A^B d\vec{l} \cdot \vec{\nabla} U = U(A) - U(B)}_{\text{Does not depend on } P\text{, just on A \& B!}} \\ \text{Non-Cons. Force: Depends on path} \end{cases}$$

- Okay, some of this is stuff you've seen before. But what if I gave you, say, a POTENTIAL ENERGY in polar coords. What is the force?

How do we evaluate the gradient in polar coords?

$$U(\rho,\phi) \Rightarrow \vec{F} = -\vec{\nabla} U(\rho,\phi) = ?$$

- Let's go back to our starting point for the gradient:

$$df = d\vec{l} \cdot (\vec{\nabla} f)$$

- Notice that I'm not referring to coordinates here! This statement is true, period, no matter what coords I use to describe things!

$$df = \frac{df}{dq_1} dq_1 + \frac{df}{dq_2} dq_2$$

Just a statement about a function of two variables.

$$d\vec{l} = h_1 dq_1 \hat{e}_1 + h_2 dq_2 \hat{e}_2$$

How we write $d\vec{l}$ in an OCS $(q_1, q_2)$ w/ scale factors $h_1$ & $h_2$, unit vectors $\hat{e}_1$ & $\hat{e}_2$

$$\vec{\nabla} f = (?)\,\hat{e}_1 + (?)\,\hat{e}_2$$

Grad $f$ is a <u>vector</u>, so it has comp. in $\hat{e}_1$ & $\hat{e}_2$ directions.

$$df = d\vec{l} \cdot \vec{\nabla} f \Rightarrow \underbrace{\boxed{\vec{\nabla} f = \frac{1}{h_1}\frac{df}{dq_1}\hat{e}_1 + \frac{1}{h_2}\frac{df}{dq_2}\hat{e}_2}}_{\text{GRADIENT IN AN OCS}}$$

- This makes sense physically. The $\hat{e}_1$ component is how $f$ changes when we move a distance $h_1 dq_1$ in that direction, and likewise for the $\hat{e}_2$ component.

- POLAR: $$\vec{\nabla} f = \frac{df}{d\rho}\hat{\rho} + \frac{1}{\rho}\frac{df}{d\phi}\hat{\phi}$$

SPC: $$\vec{\nabla} f(r,\theta,\phi) = \frac{df}{dr}\hat{r} + \frac{1}{r}\frac{df}{d\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{df}{d\phi}\hat{\phi}$$

## A Vector Differential Operator

- So we've got this thing called the gradient of $f$: $\vec{\nabla} f$. What else can we do with it?

- Before we try to answer that, let me make a suggestion. We often talk about the derivative as if it were its own thing. That is, we'll refer to '$d/dx$' as if that were one of the mathematical objects we encounter in our work.

- But what <u>is</u> $d/dx$? It's not a <u>thing</u> the same way that a number or a function is a thing, right? We might talk about the value that $df/dx$ takes for different values of $x$, but we'd never talk about the value of $d/dx$ <u>itself</u>.

- No, it's more like '+' or '−' or some other <u>operation</u>. Part of its meaning comes from the fact that it has to OPERATE on something.

- So we call $d/dx$ an "OPERATOR" b/c it is meant to operate on something. That is, part of its meaning is that we feed $d/dx$ a function $f(x)$ & it gives us a related function we call $df/dx$.

- There are many different kinds of operators, so if we're being specific we'd call $d/dx$ a DIFFERENTIAL OPERATOR.

- Likewise, we say that $\vec{\nabla}$ is a VECTOR DIFFERENTIAL OPERATOR.

- It's exactly what it's name means! We feed it functions (so its an operator) & it returns a vector's worth of related function via x, y, & z derivatives.

$$\vec{\nabla} = \hat{x}\frac{d}{dx} + \hat{y}\frac{d}{dy} + \hat{z}\frac{d}{dz}$$

- So this may seem a bit abstract, and it is, but it's useful b/c it helps us understand what else we can do w/ $\vec{\nabla}$!

- For example, what if we had a VECTOR FIELD?

$$\vec{A}(x,y,z) = A_x(x,y,z)\,\hat{x} + A_y(x,y,z)\,\hat{y} + A_z(x,y,z)\,\hat{z}$$

Is there a way that $\vec{\nabla}$ could operate on that?

- Well, sure. After all, $\vec{\nabla}$ is "vector-like." It has components like a vector, it's just that each one expects to be "fed" a function. So how would we combine it w/ a vector field? Same way we'd combine any two vectors:

$$\underbrace{\vec{\nabla}\cdot\vec{A}}_{\text{'DIVERGENCE'}} \quad \text{or} \quad \underbrace{\vec{\nabla}\times\vec{A}}_{\text{'CURL'}}$$

## THE DIVERGENCE

- The DIVERGENCE of a V.F. $\vec{A}(x,y,z)$ is a SCALAR function that we get by feeding its components to the derivatives of $\vec{\nabla}$ according to the dot product.

- So given $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$ :

$$\vec{\nabla}\cdot\vec{A} = \left(\hat{x}\frac{d}{dx} + \hat{y}\frac{d}{dy} + \hat{z}\frac{d}{dz}\right)\cdot\left(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}\right)$$

$$= \hat{x}\cdot\left(\frac{d}{dx}\left(A_x\hat{x}+A_y\hat{y}+A_z\hat{z}\right)\right) + \hat{y}\cdot\left(\frac{d}{dy}\left(A_x\hat{x}+A_y\hat{y}+A_z\hat{z}\right)\right)$$

$$+ \hat{z}\cdot\left(\frac{d}{dz}\left(A_x\hat{x}+A_y\hat{y}+A_z\hat{z}\right)\right)$$

Feed $\vec{A}$ to $\frac{d}{dx}$, etc, then dot result into $\hat{x}$, etc.

$$= \hat{x}\cdot\left(\hat{x}\frac{dA_x}{dx} + \hat{y}\frac{dA_y}{dx} + \hat{z}\frac{dA_z}{dx}\right) + \hat{y}\cdot(\dots) + \hat{z}\cdot(\dots)$$

$\hat{x}\cdot\hat{x}=1$ $\quad$ $\hat{x}\cdot\hat{y}=\hat{x}\cdot\hat{z}=0$

CAREFUL! $\frac{d}{dx}\hat{x}=0$, etc, but $\frac{d}{dq_i}\hat{e}_j \neq 0$ for some OCS!

$$= \frac{dA_x}{dx} + \frac{dA_y}{dy} + \frac{dA_z}{dz}$$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{A} = \frac{dA_x}{dx} + \frac{dA_y}{dy} + \frac{dA_z}{dz}}$$

- This is the div. in Cartesian Coords. In OCS the details will be different but the idea is the same: Derivatives of the components of $\vec{A}$ (and scale factors, in an OCS) are added together to produce a scalar function.

Ex) $\vec{A} = \underbrace{xy}_{A_x}\hat{x} + \underbrace{2y^2}_{A_y}\hat{y} + \underbrace{\frac{x^2y^2}{z^2}}_{A_z}\hat{z}$

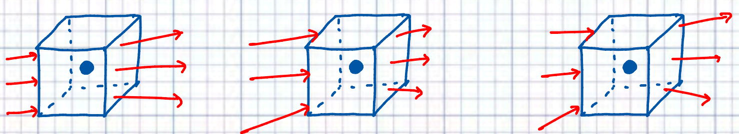
$$\vec{\nabla}\cdot\vec{A} = \frac{d}{dx}(xy) + \frac{d}{dy}(2y^2) + \frac{d}{dz}\left(\frac{x^2y^2}{z^2}\right)$$

$$= y + 4y - 2\frac{x^2y^2}{z^3}$$

$$\hookrightarrow \vec{\nabla}\cdot\vec{A} = 5y - 2\frac{x^2y^2}{z^3}$$

- Later, we'll work out how to evaluate $\vec{\nabla}\cdot\vec{A}$ in an OCS. But first, what does it <u>mean</u>?

- The divergence has a simple & really useful interpretation. Pick some pt. & imagine an infinitesimal volume $dV$ around it. If $\vec{\nabla}\cdot\vec{A} > 0$ @ that point, more $\vec{A}$ is emerging from $dV$ than going into it. If $\vec{\nabla}\cdot\vec{A} < 0$, it's the reverse. And if $\vec{\nabla}\cdot\vec{A} = 0$, then as much $\vec{A}$ is emerging as is going in:



$\vec{\nabla}\cdot\vec{A} > 0$ $\qquad$ $\vec{\nabla}\cdot\vec{A} < 0$ $\qquad$ $\vec{\nabla}\cdot\vec{A} = 0$

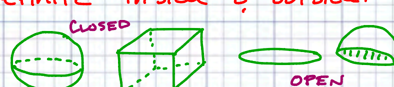
- Just like the gradient, the Divergence has its own "Fundamental Theorem." But to state it, we need to remind ourselves of a few things about volume & surface integrals.

## – INTERLUDE –

- A surface integral visits every point on a surface, multiplies the value of some function (the integrand) by an infinitesimal area $da$, and adds the result to a running total.

$$\int_S da\, f \quad \text{or} \quad \oint_S da\, f$$

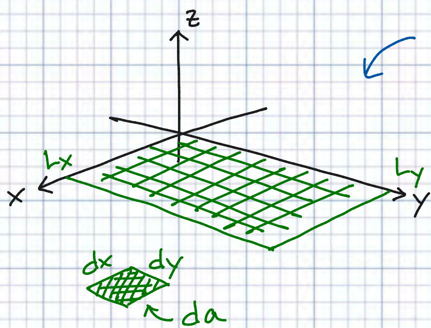
A <u>closed</u> surface has a definite inside & outside.




<u>Open</u> surfaces don't have an unambiguous inside/outside.

- You need <u>two coords</u> to describe a point on a surface, so a surface integral always involves integrating over <u>two variables</u>.

$$S:\ 0 \le x \le L_x,\ 0 \le y \le L_y$$




$$\text{Area} = \int_S da = \int_0^{L_x} dx \int_0^{L_y} dy$$

$$= \int_0^{L_x} dx \left( y \Big|_0^{L_y} \right) = \int_0^{L_x} dx\, L_y$$

$$= L_y \left( x \Big|_0^{L_x} \right) = L_x L_y \ \checkmark$$

What if we integrate $f(x,y) = xy^2$ over $S$?

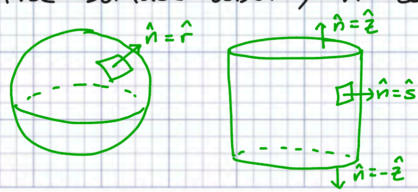
$$\int_S da\, f = \int_0^{L_x} dx \int_0^{L_y} dy\, xy^2 = \int_0^{L_x} dx \left( \tfrac{1}{3} x y^3 \Big|_0^{L_y} \right)$$

$$= \int_0^{L_x} dx\, \tfrac{1}{3} x L_y^3 = \tfrac{1}{6} x^2 L_y^3 \Big|_0^{L_x} = \tfrac{1}{6} L_x^2 L_y^3$$

- The <u>vector</u> area element $d\vec{a}$ is just $da$ on the surface, multiplied by the <u>NORMAL</u> vector:

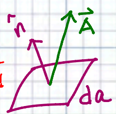
$$d\vec{a} = da\, \hat{n}$$

For a <u>closed</u> $S$, $\hat{n}$ points from <u>inside</u> to <u>outside</u>. For an <u>open</u> $S$ there are two choices & we just pick one. For the surface above, $\hat{n}$ could be $\hat{z}$ or $-\hat{z}$.



- The integral of $\vec{A}\cdot\hat{n}$ over a surface is called the <u>FLUX</u> of $\vec{A}$ through the surface. It's a measure of how much $\vec{A}$ passes <u>across</u> the surface:

Only the part of $\vec{A}$ $\perp$ to the surface 'crosses' it! $A_\perp = \hat{n}\cdot\vec{A}$ →




$$\int_S d\vec{a}\cdot\vec{A} = \int_S da\,\hat{n}\cdot\vec{A}$$

← FLUX of $\vec{A}$ across a surface $S$ is an important concept!!

- For a <u>closed</u> surface, a positive flux means a net flow of $\vec{A}$ from inside to outside. For a negative flux it's the opposite. For an open surface a positive (negative) flux means a net flow of $\vec{A}$ from one side to the other in the direction of (or opposite to) the normal.

- For a <u>VOLUME INTEGRAL</u> we visit every point inside some 3-D region, multiply the value of some function @ that point by an infinitesimal volume $dV$, & add that to a running total:

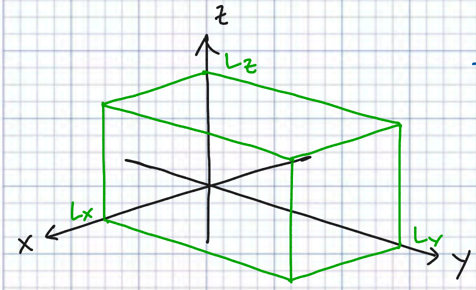
$$\int_V dV\, f$$

Volume element

integrand

region

- We need 3 coords to specify where you are in some 3-D region, so a volume integral always involves integrating over <u>three</u> variables.




$\mathcal{V}: \ 0 \le x \le L_x, \ 0 \le y \le L_y, \ 0 \le z \le L_z$

$$\underbrace{\int_{\mathcal{V}} dV}_{\text{Add up all the little volumes...}} = \int_0^{L_x} dx \int_0^{L_y} dy \int_0^{L_z} dz = \underbrace{L_x L_y L_z}_{\text{... get the total volume.}} \checkmark$$

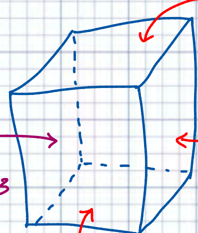
Integrate $f(x,y,z) = xyz^3$ over $\mathcal{V}$?

$$\int_{\mathcal{V}} dV\, f = \int_0^{L_x} dx \int_0^{L_y} dy \int_0^{L_z} dz\; xyz^3 = \frac{1}{16} L_x^2 L_y^2 L_z^4$$

CHECK THIS!

- For non-Cartesian surfaces & volumes we usually use a suitable OCS. Remember, scale factors relate the $dq_i$ to distances:

$q_3 = \text{const.}$
$da = h_1 h_2\, dq_1\, dq_2$



$dV = h_1 h_2 h_3\, dq_1\, dq_2\, dq_3$

$q_2 = \text{const.}$
$da = h_1 h_3\, dq_1\, dq_3$

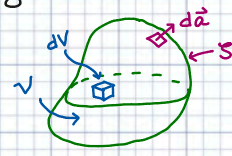
$q_1 = \text{const.}$
$da = h_2 h_3\, dq_2\, dq_3$

— END INTERLUDE —

- Now back to the divergence & its fundamental theorem. Imagine some 3-D region that we'll call $\mathcal{V}$. It's surrounded by a surface $\mathcal{S}$, and since $\mathcal{V}$ is <u>inside</u> $\mathcal{S}$ it must be that $\mathcal{S}$ is a <u>CLOSED</u> surface.

- If we add up $dV\, \vec{\nabla}\cdot\vec{A}$ for every little $dV$ inside $\mathcal{V}$, the result is the FLUX of $\vec{A}$ through $\mathcal{S}$:

$$\int_{\mathcal{V}} dV\, \vec{\nabla}\cdot\vec{A} = \oint_{\mathcal{S}} d\vec{a}\cdot\vec{A}$$

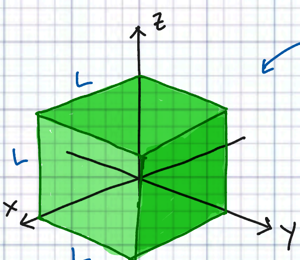



This is known as the DIVERGENCE THEOREM.

<u>Ex</u> $\vec{A} = x^2 y\,\hat{x} + zy^2\,\hat{y} + xyz\,\hat{z}$

$\mathcal{V}$ : Cube w/ $0 \le x \le L,\ 0 \le y \le L,\ 0 \le z \le L$

Show that the Div. Thm. is true for this example.

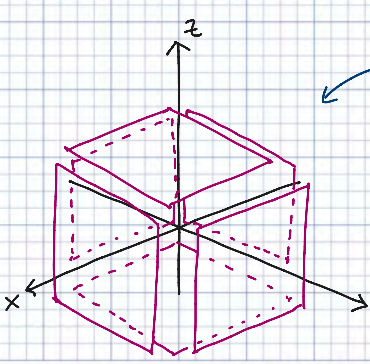



$dV = dx\,dy\,dz$

$\vec{\nabla}\cdot\vec{A} = 2xy + 2yz + xy$

$= 3xy + 2yz$

$\hookrightarrow \int_{\mathcal{V}} dV\, \vec{\nabla}\cdot\vec{A} = \int_0^L dx \int_0^L dy \int_0^L dz\, (3xy + 2yz) = \frac{5}{4} L^5$




Closed surface surrounding $\mathcal{V}$ consists of 6 squares:

(1) $x = 0$, $0 \le y,z \le L$, $d\vec{a} = dy\,dz\,(-\hat{x})$

(2) $x = L$, $0 \le y,z \le L$, $d\vec{a} = dy\,dz\,\hat{x}$

(3) $y = 0$, $0 \le x,z \le L$, $d\vec{a} = dx\,dz\,(-\hat{y})$

(4) $y = L$, $0 \le x,z \le L$, $d\vec{a} = dx\,dz\,\hat{y}$

(5) $z = 0$, $0 \le x,y \le L$, $d\vec{a} = dx\,dy\,(-\hat{z})$

(6) $z = L$, $0 \le x,y \le L$, $d\vec{a} = dx\,dy\,\hat{z}$

Flux through side 1 @ $x = 0$

$$\left.\begin{aligned} d\vec{a} &= dy\,dz\,(-\hat{x}) \\ \vec{A}(0,y,z) &= 0\,\hat{x} + zy^2\,\hat{y} + 0\,\hat{z} \end{aligned}\right\} d\vec{a}\cdot\vec{A} = 0 \text{ on side } 1$$

Flux through side 2 @ $x = L$

$$\left.\begin{aligned} d\vec{a} &= dy\,dz\,\hat{x} \\ \vec{A}(L,y,z) &= L^2 y\,\hat{x} + zy^2\,\hat{y} + Lyz\,\hat{z} \end{aligned}\right\} \begin{aligned} &d\vec{a}\cdot\vec{A} = L^2 y\,dy\,dz \\ &\text{on side } 2 \end{aligned}$$

$$\int_2 d\vec{a}\cdot\vec{A} = \int_0^L dz \int_0^L dy\; L^2 y = L^2 \int_0^L dz \left(\tfrac{1}{2} y^2\Big|_0^L\right) = \tfrac{1}{2} L^4 \int_0^L dz$$

$$= \tfrac{1}{2} L^5$$

Flux through side 3 @ $y = 0$

$$\left.\begin{aligned} d\vec{a} &= dx\,dz\,(-\hat{y}) \\ \vec{A}(x,0,z) &= 0\,\hat{x} + 0\,\hat{y} + 0\,\hat{z} \end{aligned}\right\} d\vec{a}\cdot\vec{A} = 0 \Rightarrow \text{No Flux}$$

Flux through side 4 @ $y = L$

$$\left.\begin{aligned} d\vec{a} &= dx\,dz\,\hat{y} \\ \vec{A}(x,L,z) &= Lx^2\,\hat{x} + L^2 z\,\hat{y} + Lxz\,\hat{z} \end{aligned}\right\} \begin{aligned} &d\vec{a}\cdot\vec{A} = L^2 z\,dx\,dz \\ &\text{on side } 4 \end{aligned}$$

$$\int_4 d\vec{a}\cdot\vec{A} = \tfrac{1}{2} L^5 \leftarrow \text{CHECK!}$$

Flux through side 5 @ $z = 0$ ← Also gives zero. Check this!

Flux through side 6 @ $z = L$

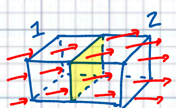
$$\left.\begin{aligned} d\vec{a} &= dx\,dy\,\hat{z} \\ \vec{A}(x,y,L) &= x^2 y\,\hat{x} + y^2 L\,\hat{y} + Lxy\,\hat{z} \end{aligned}\right\} d\vec{a}\cdot\vec{A} = Lxy\,dx\,dy \text{ on side } 6$$

$$\int_6 d\vec{a}\cdot\vec{A} = L\int_0^L dx \int_0^L dy\; xy = L \times \left(\tfrac{1}{2}L^2\right)\left(\tfrac{1}{2}L^2\right) = \tfrac{1}{4} L^5 \leftarrow \text{CHECK!}$$

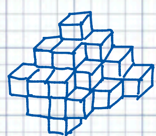
TOTAL FLUX: $\tfrac{1}{2} L^5 + \tfrac{1}{2} L^5 + \tfrac{1}{4} L^5 = \tfrac{5}{4} L^5$ ✓

- Why is this true? Recall our 'meaning' for the divergence, how it tells us whether more $\vec{A}$ is coming out of or going into an infinitesimal volume.

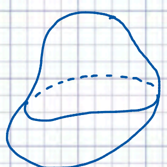
- Now stick two together, and add up $dV\, \vec{\nabla}\cdot\vec{A}$ for each. Then add a 3rd, a 4th, etc.




1 & 2 share a face. Any $\vec{A}$ passing through that face out of 1 then passes into 2. Those fluxes cancel.



All the fluxes involving common faces cancel. So all that's left is fluxes through the outer faces.



Add enough of these $dV$'s & you can build any volume $V$ that you want. And the surface around it is just the 'outer faces' of all the little cubes you used. So:

$$\int_V dV\, \vec{\nabla}\cdot\vec{A} = \oint_S d\vec{a}\cdot\vec{A}$$

- For an example like the one we just did – a cube – it made sense to work in Cartesian coords. But what if we are working with something like a cylinder? Or what if we need to evaluate the divergence of a vector given in SPC. How does the divergence work in a general OCS?

- Earlier, we saw that '$\vec{\nabla}$' in an OCS was given by

$$\vec{\nabla} = \hat{e}_1 \frac{1}{h_1}\frac{d}{dq_1} + \hat{e}_2 \frac{1}{h_2}\frac{d}{dq_2} + \hat{e}_3 \frac{1}{h_3}\frac{d}{dq_3}$$

- How does this act on something like $\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$?

- The important thing to remember is that $\frac{d}{dq_i}\hat{e}_j$ may not be zero, like it is in Cartesian coords.

- To keep things simple let's work in 2-D. We want to eval:

$$\left(\hat{e}_1 \frac{1}{h_1}\frac{d}{dq_1} + \hat{e}_2 \frac{1}{h_2}\frac{d}{dq_2}\right)\cdot\left(A_1\hat{e}_1 + A_2\hat{e}_2\right)$$

$$= \hat{e}_1\cdot\left(\frac{1}{h_1}\frac{dA_1}{dq_1}\hat{e}_1 + \frac{1}{h_1}A_1\frac{d\hat{e}_1}{dq_1} + \frac{1}{h_1}\frac{dA_2}{dq_1}\hat{e}_2 + \frac{1}{h_1}A_2\frac{d\hat{e}_2}{dq_1}\right)$$

$$+\hat{e}_2\cdot\left(\frac{1}{h_2}\frac{dA_1}{dq_2}\hat{e}_1 + \frac{1}{h_2}A_1\frac{d\hat{e}_1}{dq_2} + \frac{1}{h_2}\frac{dA_2}{dq_2}\hat{e}_2 + \frac{1}{h_2}A_2\frac{d\hat{e}_2}{dq_2}\right)$$

- We know $\hat{e}_1\cdot\hat{e}_1 = \hat{e}_2\cdot\hat{e}_2 = 1$, and $\hat{e}_1\cdot\hat{e}_2 = 0$. But what about $\hat{e}_1$ or $\hat{e}_2$ dotted into something like $d\hat{e}_1/dq_2$?

- First, we can show pretty quickly that something like $\hat{e}_1\cdot(d\hat{e}_1/dq_1) = 0$.

$$\hat{e}_1\cdot\frac{d\hat{e}_1}{dq_1} = \underbrace{\frac{1}{2}\frac{d}{dq_1}\left(\hat{e}_1\cdot\hat{e}_1\right)}_{\text{Check: } = \frac{1}{2}\left(\frac{d\hat{e}_1}{dq_1}\cdot\hat{e}_1 + \hat{e}_1\cdot\frac{d\hat{e}_1}{dq_1}\right) = \hat{e}_1\cdot\frac{d\hat{e}_1}{dq_1}\ \checkmark} = \frac{1}{2}\frac{d}{dq_1}\overbrace{(1)}^{\hat{e}_1\cdot\hat{e}_1=1} = 0$$

Der. of a constant is zero.

And the same goes for $\hat{e}_2\cdot(d\hat{e}_2/dq_2)$.

- Now what about terms like $\hat{e}_1\cdot(d\hat{e}_2/dq_1)$?

$$\hat{e}_1\cdot\frac{d\hat{e}_2}{dq_1} = \hat{e}_1\cdot\frac{d}{dq_1}\left(\frac{1}{h_2}\frac{d\vec{r}}{dq_2}\right) = \hat{e}_1\cdot\left(-\frac{1}{(h_2)^2}\frac{dh_2}{dq_1}\frac{d\vec{r}}{dq_2} + \frac{1}{h_2}\frac{d^2\vec{r}}{dq_1 dq_2}\right)$$

$$= \hat{e}_1\cdot\left(-\frac{1}{h_2}\frac{dh_2}{dq_1}\hat{e}_2 + \frac{1}{h_2}\frac{d}{dq_2}\left(\frac{d\vec{r}}{dq_1}\right)\right)$$

Order doesn't matter!

$$= -\frac{1}{h_2}\frac{dh_2}{dq_1}\cancelto{0}{\hat{e}_1\cdot\hat{e}_2} + \frac{1}{h_2}\hat{e}_1\cdot\frac{d}{dq_2}\left(h_1\hat{e}_1\right) \qquad \frac{d\vec{r}}{dq_1} = h_1\hat{e}_1$$

$$= \frac{1}{h_2}\frac{dh_1}{dq_2}\cancelto{1}{\hat{e}_1\cdot\hat{e}_1} + \frac{h_1}{h_2}\cancelto{0}{\hat{e}_1\cdot\frac{d\hat{e}_1}{dq_2}}$$

As above!

$$= \frac{1}{h_2}\frac{dh_1}{dq_2}$$

- So we get

$$\hat{e}_1 \cdot \frac{d\hat{e}_2}{dq_1} = \frac{1}{h_2}\frac{dh_1}{dq_2} \qquad \hat{e}_2 \cdot \frac{d\hat{e}_1}{dq_2} = \frac{1}{h_1}\frac{dh_2}{dq_1}$$

- And now we know how to evaluate all the terms in $\vec{\nabla}\cdot\vec{A}$:

$$\left(\hat{e}_1 \frac{1}{h_1}\frac{d}{dq_1} + \hat{e}_2 \frac{1}{h_2}\frac{d}{dq_2}\right)\cdot\left(A_1\hat{e}_1 + A_2\hat{e}_2\right)$$

$$= \hat{e}_1 \cdot \left(\frac{1}{h_1}\frac{dA_1}{dq_1}\hat{e}_1 + \frac{1}{h_1}A_1\cancel{\frac{d\hat{e}_1}{dq_1}} + \frac{1}{h_1}\frac{dA_2}{dq_1}\cancel{\hat{e}_2} + \frac{1}{h_1}A_2\frac{d\hat{e}_2}{dq_1}\right)$$

$$+\hat{e}_2 \cdot \left(\frac{1}{h_2}\frac{dA_1}{dq_2}\cancel{\hat{e}_1} + \frac{1}{h_2}A_1\frac{d\hat{e}_1}{dq_2} + \frac{1}{h_2}\frac{dA_2}{dq_2}\hat{e}_2 + \frac{1}{h_2}A_2\cancel{\frac{d\hat{e}_2}{dq_2}}\right)$$

$$= \frac{1}{h_1}\frac{dA_1}{dq_1} + \frac{1}{h_1}A_2\frac{1}{h_2}\frac{dh_1}{dq_2} + \frac{1}{h_2}A_1\frac{1}{h_1}\frac{dh_2}{dq_1} + \frac{1}{h_2}\frac{dA_2}{dq_2}$$

$$= \frac{1}{h_1 h_2}\frac{d}{dq_1}\left(h_2 A_1\right) + \frac{1}{h_1 h_2}\frac{d}{dq_2}\left(h_1 A_2\right)$$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{A} = \frac{1}{h_1 h_2}\frac{d}{dq_1}\left(h_2 A_1\right) + \frac{1}{h_1 h_2}\frac{d}{dq_2}\left(h_1 A_2\right)}$$

- So, for instance, in POLAR COORDS w/ $h_\rho = 1$ & $h_\phi = \rho$ we get:

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho A_\rho\right) + \frac{1}{\rho}\frac{d}{d\phi}\left(A_\phi\right)$$

$$= \frac{dA_\rho}{d\rho} + \frac{1}{\rho}A_\rho + \frac{1}{\rho}\frac{dA_\phi}{d\phi}$$

- Notice that this <u>is not</u> $dA_\rho/d\rho + dA_\phi/d\phi$ !

- Just like the gradient, the divergence incorporates the scale factors of an OCS. You can't just add up $dA_i/dq_i$ as in Cartesian coords!

- In 3-D it's the same idea, though we have to worry about both $\hat{e}_1 \cdot d\hat{e}_2/dq_1$ & $\hat{e}_1 \cdot d\hat{e}_3/dq_1$, etc. The final result is:

$$\boxed{\vec{\nabla} \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left( \frac{d}{dq_1}(A_1 h_2 h_3) + \frac{d}{dq_2}(A_2 h_1 h_3) + \frac{d}{dq_3}(A_3 h_1 h_2) \right)}$$

- To obtain this we used the following results, which we'll use again later:

$$\underbrace{\hat{e}_i \cdot \frac{d\hat{e}_j}{dq_i}}_{\text{Assumes } i \neq j!} = \frac{1}{h_j} \frac{dh_i}{dq_j} \qquad \underbrace{\hat{e}_i \cdot \frac{d\hat{e}_i}{dq_j}}_{\text{True for } j = i \text{ or } j \neq i} = 0$$

Ex| SPC $r, \theta, \phi$ w/ $h_r = 1$, $h_\theta = r$, $h_\phi = r\sin\theta$

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r^2 \sin\theta} \left( \underbrace{\frac{d}{dr}(r^2 \sin\theta A_r)}_{\sin\theta \frac{d}{dr}(r^2 A_r)} + \underbrace{\frac{d}{d\theta}(r \sin\theta A_\theta)}_{r \frac{d}{d\theta}(\sin\theta A_\theta)} + \underbrace{\frac{d}{d\phi}(r A_\phi)}_{r \frac{dA_\phi}{d\phi}} \right)$$

$$\hookrightarrow \vec{\nabla} \cdot \vec{A} = \frac{1}{r^2} \frac{d}{dr}(r^2 A_r) + \frac{1}{r\sin\theta} \frac{d}{d\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta} \frac{dA_\phi}{d\phi}$$

CHECK: $\vec{A} = 4\hat{r} + 0\hat{\theta} + 0\hat{\phi}$

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r^2} \frac{d}{dr}(4r^2) = \frac{8}{r}$$

$\hat{r} = \dfrac{x\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{x^2+y^2+z^2}}$ $\longleftarrow$ Use $\vec{\nabla} \cdot \vec{A}$ in Cartesian $\longrightarrow$ $\vec{\nabla} \cdot \vec{A} = \dfrac{8}{\sqrt{x^2+y^2+z^2}}$ ✓

# THE CURL

- The CURL of a V.F. $\vec{A}(x,y,z)$ is a new V.F. that we get by feeding the components of $\vec{A}$ to $\vec{\nabla}$ according to the rules for taking a cross product.

- Recall that

$$\vec{B}\times\vec{A} = (B_y A_z - B_z A_y)\,\hat{x} + (B_z A_x - B_x A_z)\,\hat{y} + (B_x A_y - B_y A_x)\,\hat{z}$$

- So in Cartesian coords the CURL of $\vec{A}$ is

$$\boxed{\vec{\nabla}\times\vec{A} = \left(\frac{dA_z}{dy} - \frac{dA_y}{dz}\right)\hat{x} + \left(\frac{dA_x}{dz} - \frac{dA_z}{dx}\right)\hat{y} + \left(\frac{dA_y}{dx} - \frac{dA_x}{dy}\right)\hat{z}}$$

Ex| $\vec{A} = -y\,\hat{x} + x\,\hat{y} + 0\,\hat{z}$

$$\vec{\nabla}\times\vec{A} = \left(\frac{d}{dy}(0) - \underbrace{\cancel{\frac{d}{dz}(x)}}_{0}\right)\hat{x} + \left(\underbrace{\cancel{\frac{d}{dz}(-y)}}_{0} - \frac{d}{dx}(0)\right)\hat{y} + \left(\underbrace{\cancel{\frac{d}{dx}(x)}}_{1} - \underbrace{\cancel{\frac{d}{dy}(-y)}}_{-1}\right)\hat{z} = 2\,\hat{z}$$

- As we did w/ the divergence, we'll work out how to evaluate the curl in a general OCS. But first let's talk about what it means.

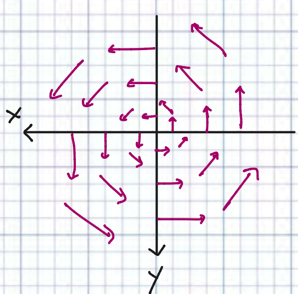
- The divergence of a V.F. @ a point tells us whether more V.F. is coming out of or going into a tiny region around that pt. The CURL, on the other hand, tells us if the vector has any "rotation" or "curliness" around that point.

- Remember: $\vec{\nabla}\times\vec{A}$ is a <u>vector</u>. In our example it was $2\hat{z}$.

- The direction of this vector gives us an axis, and its sign & magnitude tell us the 'amount' & direction of the vector field's curliness around that axis, @ that point.

- If you point your thumb along the axis, the vector field has some "rotation" or "circulation" around the axis in the direction of your curled fingers @ that point.

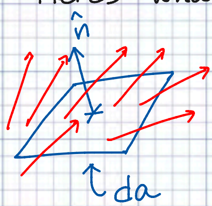
- Consider the example we just worked out. Looking down from above (the $z$-axis is coming out of the page) the v.f. looks like



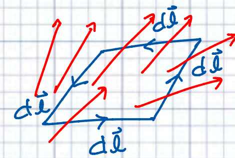

Your thumb sticks out of the page, and @ any point the net 'curliness' of V.F. around your thumb @ that point is CCW.

- Now let's be a bit more precise. Imagine a tiny (infinitesimal) patch of area around a point, with area $da$ & normal (perpendicular) direction $\hat{n}$. Then $d\vec{a}\cdot(\vec{\nabla}\times\vec{A})$ is basically $d\vec{l}\cdot\vec{A}$ added up around the perimeter of that little patch.

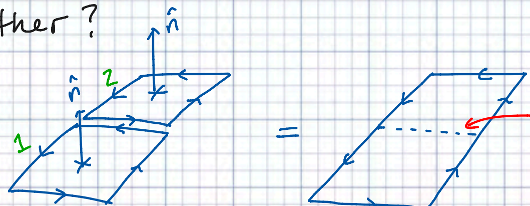
- Heres what I mean




$$\underbrace{d\vec{a}\cdot(\vec{\nabla}\times\vec{A})}_{\text{Flux of } \vec{\nabla}\times\vec{A} \text{ across tiny patch of surface}} \sim \underbrace{\sum d\vec{l}\cdot\vec{A}}_{\text{Add up } d\vec{l}\cdot\vec{A} \text{ around perimeter.}}$$

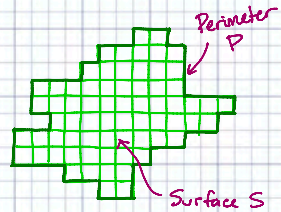



- Now what if we stick two of these little patches together?




The $d\vec{l}\cdot\vec{A}$ along the common edge cancels out b/c $d\vec{l}_2 = -d\vec{l}_1$ there!

So if we add up $d\vec{a}\cdot(\vec{\nabla}\times\vec{A})$ for two adjacent patches, get $\sum d\vec{l}\cdot\vec{A}$ around their outer perimeter.

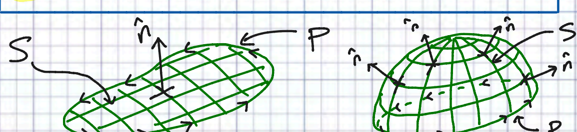



- As we add more & more we can build up any surface. Adding together $d\vec{a}\cdot(\vec{\nabla}\times\vec{A})$ for each tiny patch gives us the flux of $\vec{\nabla}\times\vec{A}$ across the surface, which is equal to $d\vec{l}\cdot\vec{A}$ added up around its perimeter!

$$\boxed{\int_S d\vec{a}\cdot(\vec{\nabla}\times\vec{A}) = \oint_P d\vec{l}\cdot\vec{A}}$$

STOKES'S THEOREM

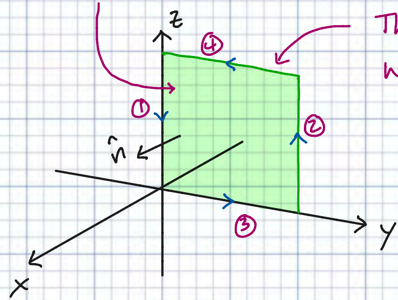
Perimeter of S is CLOSED.



Ex $\vec{A} = xyz\,\hat{x} + y^2 z\,\hat{y} + (z^2 x + 3z^3)\,\hat{z}$

$\vec{\nabla}\times\vec{A} = -y^2\hat{x} + (xy - z^2)\,\hat{y} - xz\,\hat{z}$ ← CHECK!

S: $x = 0$ , $0 \le y, z \le L$




The perimeter of S is the square w/ sides

(1) $x=0, y=0, 0\le z\le L$

(2) $x=0, y=L, 0\le z\le L$

(3) $x=0, z=0, 0\le y\le L$

(4) $x=0, z=L, 0\le y\le L$

(1) Flux of $\vec{\nabla}\times\vec{A}$ across S?

$\hat{n} = \hat{x}$ & $\vec{\nabla}\times\vec{A} = -y^2\hat{x} + (\cancel{x}^{0}y - z^2)\,\hat{y} - \cancel{x}^{0}z\,\hat{z}$

$\rightarrow \hat{n}\cdot(\vec{\nabla}\times\vec{A}) = -y^2$ ← Surface has $x=0$, but comp. we want only depends on y.

$\rightarrow d\vec{a}\cdot(\vec{\nabla}\times\vec{A}) = -y^2\,dy\,dz$

$$\int_S d\vec{a}\cdot(\vec{\nabla}\times\vec{A}) = -\int_0^L dz\int_0^L dy\; y^2 = -\int_0^L dz\left(\frac{1}{3}y^3\Big|_0^L\right) = -\int_0^L dz\,\frac{L^3}{3}$$

$$= -\frac{1}{3}L^4$$ ← $\hat{n}$ is $\hat{x}$, so negative flux means crossing in $-\hat{x}$ direction.

(2) Integral of $d\vec{l}\cdot\vec{A}$ around perimeter of S?

(1) $x=0, y=0, 0\le z\le L \Rightarrow d\vec{l} = dz\,\hat{z}$

$\vec{A}(0,0,z) = 3z^3\,\hat{z} \Rightarrow d\vec{l}\cdot\vec{A} = 3z^3 dz$ on side 1

$$\int_0^L dz\, 3z^3 = \frac{3}{4}z^4\Big|_0^L = \frac{3}{4}L^4$$

(2) $x = 0, \ y = L, \ 0 \le z \le L$

$$\left.\begin{aligned} d\vec{l} &= -\,dz\,\hat{z} \\ \vec{A}(0,L,z) &= L^2 z\,\hat{y} + 3z^3\,\hat{z} \end{aligned}\right\} d\vec{l}\cdot\vec{A} = -3z^3\,dz$$

$$\int_0^L dz\,(-3z^3) = -\frac{3}{4}L^4$$

Notice the dir. of $d\vec{l}$ on sides 2 & 4, to move us around the perimeter in the right direction.

(3) $x = 0, \ z = 0, \ 0 \le y \le L$

$$\left.\begin{aligned} d\vec{l} &= dy\,\hat{y} \\ \vec{A}(0,y,0) &= 0 \end{aligned}\right\} d\vec{l}\cdot\vec{A} = 0$$

(4) $x = 0, \ z = L, \ 0 \le y \le L$

$$\left.\begin{aligned} d\vec{l} &= -dy\,\hat{y} \\ \vec{A}(0,y,L) &= Ly^2\,\hat{y} + 3L^3\,\hat{z} \end{aligned}\right\} d\vec{l}\cdot\vec{A} = -Ly^2$$

$$\int_0^L dy\,(-Ly^2) = -\frac{1}{3}Ly^3\Big|_0^L = -\frac{1}{3}L^4$$

$$\hookrightarrow \oint_P d\vec{l}\cdot\vec{A} = \frac{3}{4}L^4 - \frac{3}{4}L^4 + 0 - \frac{1}{3}L^4 = -\frac{1}{3}L^4 \ \checkmark$$

- A special case of STOKES'S THM is <u>GREEN'S THM</u>.
- Consider a V.F. $\vec{A} = A_x\,\hat{x} + A_y\,\hat{y} + 0\,\hat{z}$ & let $S$ be a surface in the x-y plane, so $\hat{n} = \hat{z}$.

$$\vec{\nabla}\times\vec{A} = -\frac{dA_y}{dz}\,\hat{x} + \frac{dA_x}{dz}\,\hat{y} + \left(\frac{dA_y}{dx} - \frac{dA_x}{dy}\right)\hat{z}$$

$$\hookrightarrow \int_S d\vec{a}\cdot(\vec{\nabla}\times\vec{A}) = \int_S da\left(\frac{dA_y}{dx} - \frac{dA_x}{dy}\right)$$

- Since S is in the plane, $d\vec{l} = dx\,\hat{x} + dy\,\hat{y}$ on its perimeter

$$d\vec{l}\cdot\vec{A} = A_x\,dx + A_y\,dy \rightarrow \oint_P d\vec{l}\cdot\vec{A} = \int_P (dx\,A_x + dy\,A_y)$$

$$\hookrightarrow \int_P (dx\,A_x + dy\,A_y) = \int_S dx\,dy\left(\frac{dA_y}{dx} - \frac{dA_x}{dy}\right)$$

Green's Thm as a mathematician would write it, which is just an application of Stokes's Thm!

- Now, the surface & its perimeter in our last example were easy to describe in Cartesian coords. But what if we're considering a surface or VF that is most easily described in some other OCS? How do we take the curl in a general OCS?

- That is, what do we get when we evaluate:

$$\left(\hat{e}_1\frac{1}{h_1}\frac{d}{dq_1} + \hat{e}_2\frac{1}{h_2}\frac{d}{dq_2} + \hat{e}_3\frac{1}{h_3}\frac{d}{dq_3}\right)\times\left(\hat{e}_1A_1 + \hat{e}_2A_2 + \hat{e}_3A_3\right) = ?$$

- As w/ the div, the main thing to watch out for is the derivatives acting on the unit vectors. Rather than working this one out in detail, we'll quote the final result.

- First, it's important to make sure we present our OCS in the correct order. The unit vectors should satisfy:

$$\hat{e}_1\times\hat{e}_2 = \hat{e}_3 \qquad \hat{e}_2\times\hat{e}_3 = \hat{e}_1 \qquad \hat{e}_3\times\hat{e}_1 = \hat{e}_2$$

- Then the curl of $\vec{A}$ in an OCS is:

$$\boxed{\begin{aligned}\vec{\nabla}\times\vec{A} = \;& \hat{e}_1 \frac{1}{h_2 h_3}\left(\frac{d}{dq_2}(h_3 A_3) - \frac{d}{dq_3}(h_2 A_2)\right) \\ &+ \hat{e}_2 \frac{1}{h_1 h_3}\left(\frac{d}{dq_3}(h_1 A_1) - \frac{d}{dq_1}(h_3 A_3)\right) \\ &+ \hat{e}_3 \frac{1}{h_1 h_2}\left(\frac{d}{dq_1}(h_2 A_2) - \frac{d}{dq_2}(h_1 A_1)\right)\end{aligned}}$$

- (This result & the expressions for div, grad, & the Laplacian all have a geometric explanation that's much nicer than you'd guess from a 'brute force' derivation, but we don't have time to go into that level of detail.)

- Another common way of writing this involves a determinant:

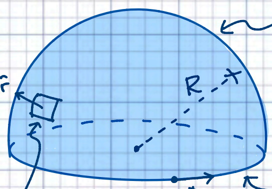
$$\vec{\nabla}\times\vec{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} \hat{e}_1 h_1 & \hat{e}_2 h_2 & \hat{e}_3 h_3 \\ \frac{d}{dq_1} & \frac{d}{dq_2} & \frac{d}{dq_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

Ex| In spherical coords the scale factors are $h_r = 1$, $h_\theta = r$, $h_\phi = r\sin\theta$. So:

$$\begin{aligned}\vec{\nabla}\times\vec{A} = \;& \hat{r}\frac{1}{\cancel{r}^2\sin\theta}\left(\frac{d}{d\theta}(\cancel{r}\sin\theta A_\phi) - \frac{d}{d\phi}(\cancel{r}A_\theta)\right) \\ &+ \hat{\theta}\frac{1}{r\sin\theta}\left(\frac{d}{d\phi}(A_r) - \frac{d}{dr}(r\sin\theta A_\phi)\right) \\ &+ \hat{\phi}\,\frac{1}{r}\left(\frac{d}{dr}(rA_\theta) - \frac{d}{d\theta}(A_r)\right)\end{aligned}$$

$$\Rightarrow \vec{\nabla}\times\vec{A} = \frac{1}{r\sin\theta}\left(\frac{d(\sin\theta A_\phi)}{d\theta} - \frac{dA_\theta}{d\phi}\right)\hat{r} + \left(\frac{1}{r\sin\theta}\frac{dA_r}{d\phi} - \frac{1}{r}\frac{d(rA_\phi)}{dr}\right)\hat{\theta} + \frac{1}{r}\left(\frac{d(rA_\theta)}{dr} - \frac{dA_r}{d\theta}\right)\hat{\phi}$$

- Let's use this to work out an example of Stokes's Thm in Spherical Polar Coordinates. For the v.f. we'll use $\vec{A} = 0\,\hat{r} + 0\,\hat{\theta} + 1\,\hat{\phi}$, & for the surface we'll use the top half (upper hemisphere) of a sphere of radius R, centered @ the origin.




$d\vec{a} = R^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$

S: Surface is $\underbrace{r = R}_{\text{All pts same dist. R from origin}}$, $\underbrace{\overset{\text{N.P.}}{0} \le \theta \le \overset{\text{EQUATOR}}{\pi/2}}_{\text{Top half of sphere}}$, $0 \le \phi < 2\pi$

P: Perimeter is the equator: $r = R$, $\theta = \pi/2$, $0 \le \phi < 2\pi$

$d\vec{l} = R \sin\frac{\pi}{2}\, d\phi\, \hat{\phi} = R\, d\phi\, \hat{\phi}$

$$\vec{\nabla} \times \vec{A} = \frac{1}{r\sin\theta}\left(\overbrace{\frac{d}{d\theta}(\sin\theta \cdot 1)}^{\cos\theta} - \cancelto{0}{\frac{d}{d\phi}(0)}\right)\hat{r} + \frac{1}{r}\left(\cancelto{0}{\frac{1}{\sin\theta}\frac{d}{d\phi}(0)} - \overbrace{\frac{d}{dr}(r \cdot 1)}^{1}\right)\hat{\theta}$$

$$+ \frac{1}{r}\left(\cancelto{0}{\frac{d}{dr}(r \cdot 0)} - \cancelto{0}{\frac{d}{d\theta}(0)}\right)\hat{\phi}$$

$$\vec{\nabla} \times \vec{A} = \frac{1}{r\sin\theta}\cos\theta\,\hat{r} - \frac{1}{r}\,\hat{\theta} + 0\,\hat{\phi}$$

$r = R$ on surface! $\hat{r}\cdot\hat{\theta} = 0$

$$\int_S d\vec{a} \cdot (\vec{\nabla} \times \vec{A}) = \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta\, R^2 \sin\theta\, \hat{r} \cdot \left(\frac{1}{R}\frac{\cos\theta}{\sin\theta}\,\hat{r} - \frac{1}{R}\cancel{\hat{\theta}} + 0\,\cancelto{0}{\hat{\phi}}\right)$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta\, R^{\cancel{2}} \cancel{\sin\theta}\, \frac{1}{\cancel{R}} \frac{\cos\theta}{\cancel{\sin\theta}} = \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta\, R\cos\theta$$

$$= \int_0^{2\pi} d\phi\, R\sin\theta \Big|_0^{\pi/2} = \int_0^{2\pi} d\phi\, R = 2\pi R$$

Now integrating $d\vec{l} \cdot \vec{A}$ around the perimeter:

$$\oint d\vec{l} \cdot \vec{A} = \int_0^{2\pi} d\phi\, R\,\hat{\phi} \cdot \underbrace{\left(0\,\cancel{\hat{r}} + 0\,\cancel{\hat{\theta}} + 1\,\hat{\phi}\right)}_{\text{This is a simple vector, but if it had any } r \text{ or } \theta \text{ dependence we'd be careful to set } r = R \text{ \& } \theta = \pi/2!} = \int_0^{2\pi} d\phi\, R = 2\pi R$$

$\hat{\phi}\cdot\hat{r} = 0$ $\hat{\phi}\cdot\hat{\theta} = 0$

## THE LAPLACIAN

- All your favorite physics equations involve 2nd derivatives. For example, the WAVE EQN

$$\frac{d^2y(x,t)}{dx^2} - \frac{1}{v^2}\frac{d^2y(x,t)}{dt^2} = 0$$

- Now that we've developed this notion of a multi-variable 'vector-ish' derivative, we should ask how it might be applied twice.
- There are multiple answers here, but we're going to focus on one of them.
- Suppose we have a scalar function. It could be the height $y(\rho,\phi)$ of a point on a vibrating drum head, the temperature $T(x,y,z)$ @ a point in a slab of metal, or anything.
- The GRADIENT of this function gives us a vector:

$$\vec{\nabla}T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$$

$$\vec{\nabla}y = \frac{\partial y}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial y}{\partial \phi}\hat{\phi}$$

- Now how could we take a "second" derivative? We now have a vector, so we could either take its curl or its divergence.
- But you can check pretty quickly that $\vec{\nabla}\times(\vec{\nabla}T) = 0$!

$$\vec{\nabla}\times(\vec{\nabla}T) = \hat{x}\underbrace{\left(\frac{\partial^2 T}{\partial y\partial z} - \frac{\partial^2 T}{\partial z\partial y}\right)}_{=0 \text{ by `EQUALITY OF MIXED PARTIALS'}} + \hat{y}\left(\frac{\partial^2 T}{\partial z\partial x} - \frac{\partial^2 T}{\partial x\partial z}\right) + \hat{z}\left(\frac{\partial^2 T}{\partial x\partial y} - \frac{\partial^2 T}{\partial y\partial x}\right) = 0$$

- So $\vec{\nabla} \times (\vec{\nabla} T) = 0$ no matter what $T$ is. And this is true no matter what coordinates you're using – if a vector is zero in Cartesian it can't somehow be non-zero in other coords!

- This leaves the divergence:

$$\vec{\nabla} \cdot (\vec{\nabla} T) = \frac{d^2 T}{dx^2} + \frac{d^2 T}{dy^2} + \frac{d^2 T}{dz^2}$$

- We call this the <u>LAPLACIAN</u>, after Pierre-Simon, marquis de Laplace. It is typically abbreviated as '$\nabla^2 T$', and if the Laplacian of a function is zero we call that 'Laplace's Equation':

$$\nabla^2 T = \vec{\nabla} \cdot (\vec{\nabla} T) = \frac{d^2 T}{dx^2} + \frac{d^2 T}{dy^2} + \frac{d^2 T}{dz^2} = 0$$

<u>Ex</u> Evaluate the Laplacian of $f(x,y,z) = x^2 y^3 e^{-z^2}$

$$\frac{d}{dx} f = 2x y^3 e^{-z^2} \qquad \frac{d^2 f}{dx^2} = 2 y^3 e^{-z^2}$$

$$\frac{d}{dy} f = 3x^2 y^2 e^{-z^2} \qquad \frac{d^2 f}{dy^2} = 6 x^2 y \, e^{-z^2}$$

$$\frac{d}{dz} f = -2z \, x^2 y^3 e^{-z^2} \qquad \frac{d^2 f}{dz^2} = -2 x^2 y^3 e^{-z^2} + 4 z^2 x^2 y^3 e^{-z^2}$$

$$\hookrightarrow \nabla^2 f = \left(2y^3 + 6x^2 y - 2x^2 y^3 + 4 z^2 x^2 y^3\right) e^{-z^2}$$

- But what if we want to use CPC, SPC, or some other OCS? We know how to evaluate both the gradient & divergence in an OCS, so working out the Laplacian is straightforward:

$$\vec{\nabla} f = \frac{1}{h_1}\frac{df}{dq_1}\hat{e}_1 + \frac{1}{h_2}\frac{df}{dq_2}\hat{e}_2 + \frac{1}{h_3}\frac{df}{dq_3}\hat{e}_3$$

$$\vec{\nabla}\cdot(\vec{\nabla} f) = \frac{1}{h_1h_2h_3}\left(\frac{d}{dq_1}\left(h_2h_3\frac{1}{h_1}\frac{df}{dq_1}\right) + \frac{d}{dq_2}\left(h_1h_3\frac{1}{h_2}\frac{df}{dq_2}\right) + \frac{d}{dq_3}\left(h_1h_2\frac{1}{h_3}\frac{df}{dq_3}\right)\right)$$

$$\boxed{\nabla^2 f = \frac{1}{h_1h_2h_3}\left(\frac{d}{dq_1}\left(\frac{h_2h_3}{h_1}\frac{df}{dq_1}\right) + \frac{d}{dq_2}\left(\frac{h_1h_3}{h_2}\frac{df}{dq_2}\right) + \frac{d}{dq_3}\left(\frac{h_1h_2}{h_3}\frac{df}{dq_3}\right)\right)}$$

<u>Ex</u> Write out Laplace's equation $\nabla^2\psi(r,\theta,\phi)=0$ in SPC:

$h_1 = 1 \quad h_2 = r \quad h_3 = r\sin\theta$

$$\nabla^2\psi = \frac{1}{r^2\sin\theta}\left(\frac{d}{dr}\left(r^2\sin\theta\frac{d\psi}{dr}\right) + \frac{d}{d\theta}\left(\frac{\cancel{r}\sin\theta}{\cancel{r}}\frac{d\psi}{d\theta}\right) + \frac{d}{d\phi}\left(\frac{\cancel{r}}{\cancel{r}\sin\theta}\frac{d\psi}{d\phi}\right)\right)$$

$$= \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\psi}{d\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{d^2\psi}{d\phi^2}$$

So Laplace's eqn in SPC is:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\psi}{d\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{d^2\psi}{d\phi^2} = 0$$

# ▨ Summary

## Gradient

$$\vec{\nabla} f = \frac{1}{h_1}\frac{df}{dq_1}\hat{e}_1 + \frac{1}{h_2}\frac{df}{dq_2}\hat{e}_2 + \frac{1}{h_3}\frac{df}{dq_3}\hat{e}_3$$

## Divergence

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{h_1 h_2 h_3}\left(\frac{d}{dq_1}(A_1 h_2 h_3) + \frac{d}{dq_2}(A_2 h_1 h_3) + \frac{d}{dq_3}(A_3 h_1 h_2)\right)$$

## Curl

$$\vec{\nabla}\times\vec{A} = \hat{e}_1 \frac{1}{h_2 h_3}\left(\frac{d}{dq_2}(h_3 A_3) - \frac{d}{dq_3}(h_2 A_2)\right)$$
$$+ \hat{e}_2 \frac{1}{h_1 h_3}\left(\frac{d}{dq_3}(h_1 A_1) - \frac{d}{dq_1}(h_3 A_3)\right)$$
$$+ \hat{e}_3 \frac{1}{h_1 h_2}\left(\frac{d}{dq_1}(h_2 A_2) - \frac{d}{dq_2}(h_1 A_1)\right)$$

## Laplacian

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3}\left(\frac{d}{dq_1}\left(\frac{h_2 h_3}{h_1}\frac{df}{dq_1}\right) + \frac{d}{dq_2}\left(\frac{h_1 h_3}{h_2}\frac{df}{dq_2}\right) + \frac{d}{dq_3}\left(\frac{h_1 h_2}{h_3}\frac{df}{dq_3}\right)\right)$$

## Cartesian Coordinates

$$\vec{\nabla} f = \frac{df}{dx}\hat{x} + \frac{df}{dy}\hat{y} + \frac{df}{dz}\hat{z}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{dA_x}{dx} + \frac{dA_y}{dy} + \frac{dA_z}{dz}$$

$$\vec{\nabla}\times\vec{A} = \left(\frac{dA_z}{dy} - \frac{dA_y}{dz}\right)\hat{x} + \left(\frac{dA_x}{dz} - \frac{dA_z}{dx}\right)\hat{y} + \left(\frac{dA_y}{dx} - \frac{dA_x}{dy}\right)\hat{z}$$

$$\nabla^2\psi = \frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2} + \frac{d^2\psi}{dz^2}$$

# SPHERICAL POLAR COORDS

$$(q_1, q_2, q_3) = (r, \theta, \phi) \qquad h_r = 1 \quad h_\theta = r \quad h_\phi = r\sin\theta$$

$$\vec{\nabla} f = \frac{df}{dr}\hat{r} + \frac{1}{r}\frac{df}{d\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{df}{d\phi}\hat{\phi}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{d}{d\theta}\left(\sin\theta\, A_\theta\right) + \frac{1}{r\sin\theta}\frac{dA_\phi}{d\phi}$$

$$\vec{\nabla}\times\vec{A} = \frac{1}{r\sin\theta}\left(\frac{d(\sin\theta A_\phi)}{d\theta} - \frac{dA_\theta}{d\phi}\right)\hat{r} + \left(\frac{1}{r\sin\theta}\frac{dA_r}{d\phi} - \frac{1}{r}\frac{d(rA_\phi)}{dr}\right)\hat{\theta} + \frac{1}{r}\left(\frac{d(rA_\theta)}{dr} - \frac{dA_r}{d\theta}\right)\hat{\phi}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\psi}{d\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{d^2\psi}{d\phi^2}$$

# CYLINRICAL POLAR COORDINATES

$$(q_1, q_2, q_3) = (\rho, \phi, z) \qquad h_\rho = 1 \quad h_\phi = \rho \quad h_z = 1$$

$$\vec{\nabla} f = \frac{df}{d\rho}\hat{\rho} + \frac{1}{\rho}\frac{df}{d\phi}\hat{\phi} + \frac{df}{dz}\hat{z}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho A_\rho\right) + \frac{1}{\rho}\frac{d}{d\phi}(A_\phi) + \frac{d}{dz}(A_z)$$

$$\vec{\nabla}\times\vec{A} = \left(\frac{1}{\rho}\frac{d}{d\phi}(A_z) - \frac{d}{dz}(A_\phi)\right)\hat{\rho} + \left(\frac{dA_\rho}{dz} - \frac{dA_z}{d\rho}\right)\hat{\phi}$$

$$+ \left(\frac{1}{\rho}\frac{d}{d\rho}(\rho A_\phi) - \frac{1}{\rho}\frac{d}{d\phi}(A_\rho)\right)\hat{z}$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\psi}{d\rho}\right) + \frac{1}{\rho^2}\frac{d^2\psi}{d\phi^2} + \frac{d^2\psi}{dz^2}$$